Wrocław University of Science and Technology

Soil Mechanics

-Lecture II: In-situ stresses in soil. Concept of effective stresses. Water flow.

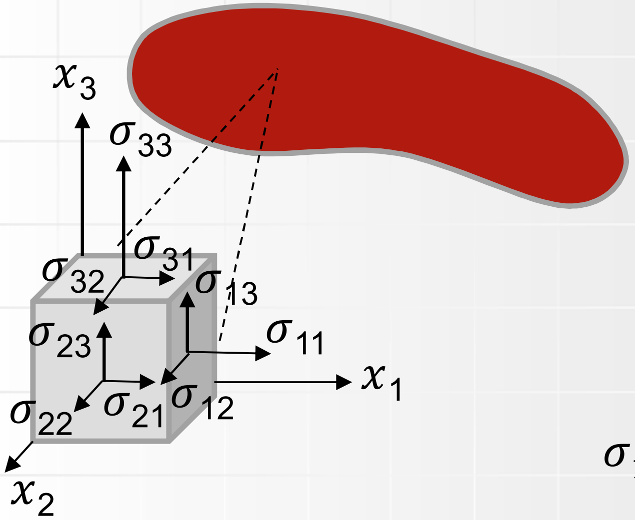
HR EXCELLENCE IN RESEARCH

European Funds Knowledge Education Development

Wrocław University of Science and Technology

European Union European Social Fund

Stress tensor



$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$I_\sigma = tr(\sigma_{ij}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3$$

$$II_\sigma = \frac{1}{2}[\mathrm{tr}(\sigma_{ij})^2 - \mathrm{tr}(\sigma_{ij}{}^2)] =$$

$$\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}\sigma_{21} - \sigma_{23}\sigma_{32} - \sigma_{13}\sigma_{31} =$$

$$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$$

$$III_\sigma = \det(\sigma_{ij}) = \sigma_1\sigma_2\sigma_3$$

$$\det(\sigma_{ij} - \lambda\delta_{ij}) = -\lambda^3 + I_\sigma\lambda^2 - II_\sigma\lambda + III_\sigma = 0$$

$$\sigma_{ij} = rim_{rjn}\sigma_{mn}$$

$$\sigma' = R\sigma R^T$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Concept of effective stress.

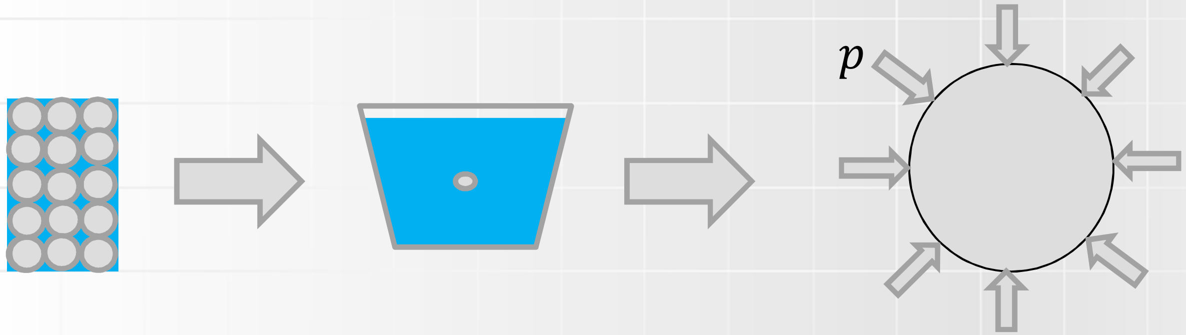
Mean hydrostatic stress tensor (volumetric, mean normal)

Stress devaitor tensor s_{ij}

$$\sigma_{ij}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}=\begin{bmatrix}\sigma_0 & 0 & 0\\ 0 & \sigma_0 & 0\\ 0 & 0 & \sigma_0\end{bmatrix}+\begin{bmatrix}\sigma_{11}-\sigma_0 & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma_0 & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_0\end{bmatrix}$$

$$\sigma_0=\frac{\mathrm{tr}(\sigma_{ij})}{3}=\frac{\sigma_{kk}}{3}=\frac{\sigma_{11}+\sigma_{22}+\sigma 33}{3}$$

$$s_{ij}=\sigma_{ij}-\frac{\sigma_{kk}}{3}\delta_{ij}$$



$$\sigma_{ij}=\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}$$

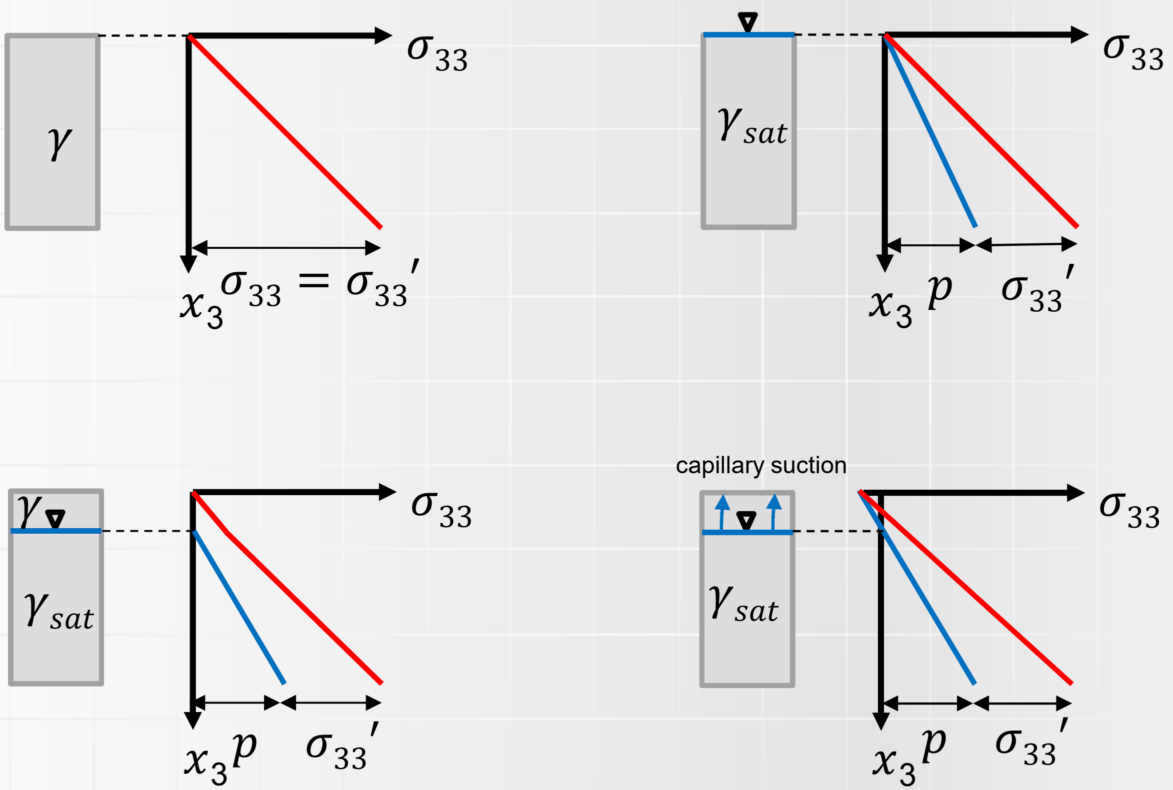
total stresses | stresses in fluid | effective stresses (stresses in soil)

$$\sigma_{i_j'}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}-\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}=\begin{bmatrix}\sigma_{11}-\sigma_0 & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma_0 & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_0\end{bmatrix}$$

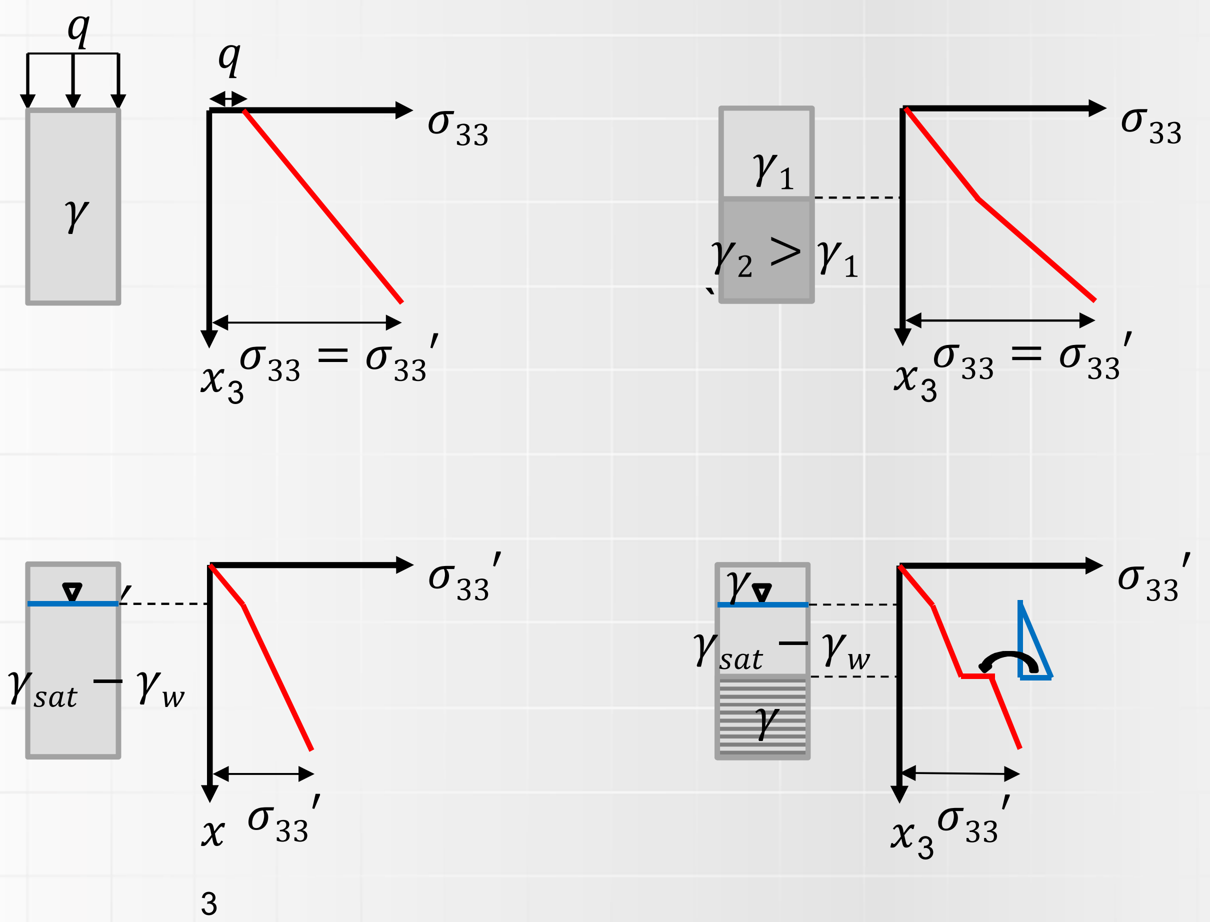
$$p\neq\sigma_0$$

In soil mechanics internal relations are often expressed in effective stresses

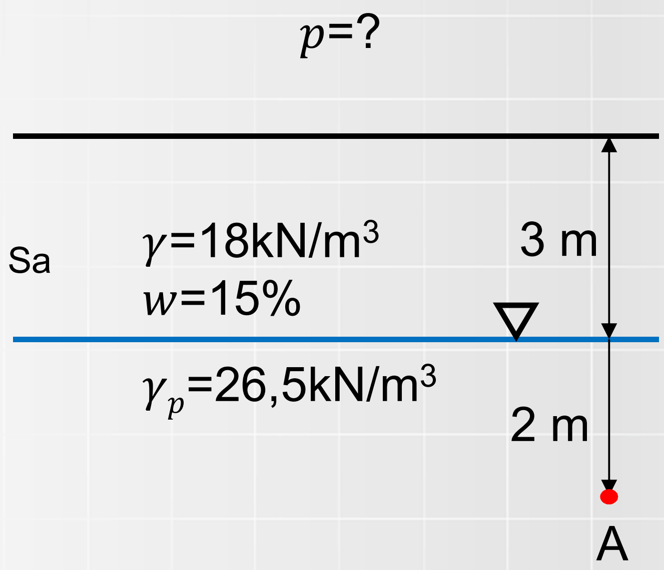
Normal vertical stress in a soil layer



Normal vertical stress in set of soil layers

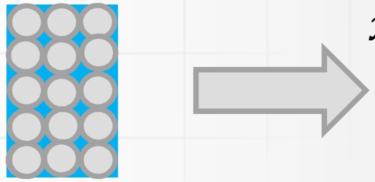


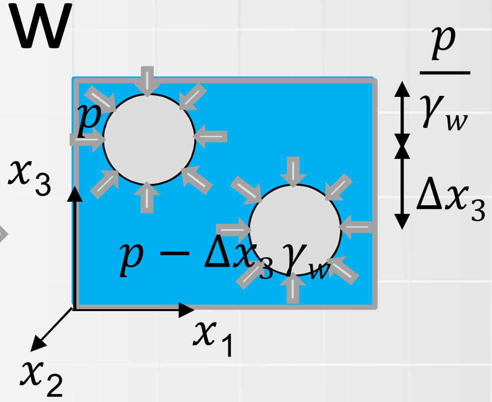
Questions



- A rubber balloon is filled with dry sand. The pressure in the pores is reduced by 5 kPa with the aid of a vacuum pump. What is the change of the total stress, and the change of the effective stress?
- Horizontal effective and total stresses knowing K_0

Darcy's Law





Static state

$$\frac{\partial p}{\partial x_1} = 0$$
$$\frac{\partial p}{\partial x_2} = 0 \quad \Rightarrow \quad p = -\gamma_w x_3 + C$$
$$\frac{\partial p}{\partial x_3} + \gamma_w = 0 \qquad (C = 0)$$

Dynamic state

$$\frac{\partial p}{\partial x_1} - f_1 = 0$$
$$\frac{\partial p}{\partial x_2} - f_2 = 0$$
$$\frac{\partial p}{\partial x_3} + \gamma_w - f_3 = 0$$

Friction force

$$f_1 = -\frac{\mu}{\kappa} q_1$$
$$f_2 = -\frac{\mu}{\kappa} q_2$$
$$f_3 = -\frac{\mu}{\kappa} q_3$$

μ-viscosity

κ-intrinsic permeability

q-specific discharge (m^3/s)/m^2

Darcy's Law

$$q_1 = -\frac{\kappa}{\mu}\frac{\partial p}{\partial x_1}$$
$$q_2 = -\frac{\kappa}{\mu}\frac{\partial p}{\partial x_2}$$
$$q_3 = -\frac{\kappa}{\mu}\left(\frac{\partial p}{\partial x_3} + \gamma_w\right)$$

Groundwater head

$$h = x_3 + \frac{p}{\gamma_w}$$
$$\frac{\partial h}{\partial x_1} = \frac{1}{\gamma_w}\frac{\partial p}{\partial x_1}$$
$$\frac{\partial h}{\partial x_2} = \frac{1}{\gamma_w}\frac{\partial p}{\partial x_2}$$
$$\frac{\partial h}{\partial x_3} = \frac{1}{\gamma_w}\left(\frac{\partial p}{\partial x_3} + \gamma_w\right)$$

$$q_1 = -k\frac{\partial h}{\partial x_1}$$
$$q_2 = -k\frac{\partial p}{\partial x_2}$$
$$q_3 = -k\frac{\partial h}{\partial x_3}$$

$$q = -k\frac{\partial h}{\partial s} = -k\, i$$

i-hydraulic gradient

$$k = \frac{\kappa\,\gamma_w}{\mu}$$ k-coefficent of permeability

Seepage force

Friction force

$$f_1 = -\frac{\mu}{\kappa} q_1$$

$$f_2 = -\frac{\mu}{\kappa} q_2$$

$$f_3 = -\frac{\mu}{\kappa} q_3$$

Darcy's Law

$$q_1 = -k\frac{\partial h}{\partial x_1}$$

$$q_2 = -k\frac{\partial p}{\partial x_2}$$

$$q_3 = -k\frac{\partial p}{\partial x_3}$$

$$f_1 = \gamma_w \frac{\partial h}{\partial x_1}$$

$$f_2 = \gamma_w \frac{\partial h}{\partial x_2}$$

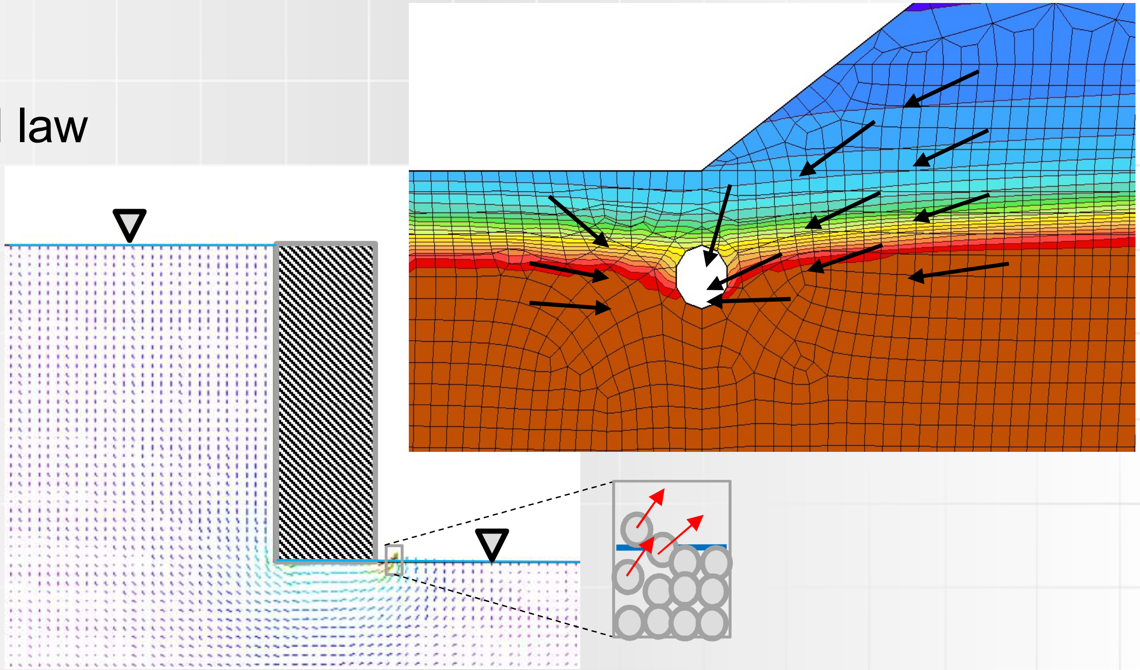
$$f_3 = \gamma_w \frac{\partial h}{\partial x_3}$$

According to Newton third law

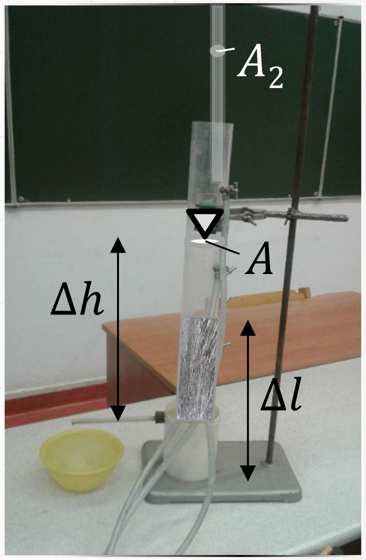
$$j_1 = -\gamma_w \frac{\partial h}{\partial x_1}$$

$$j_2 = -\gamma_w \frac{\partial h}{\partial x_2}$$

$$j_3 = -\gamma_w \frac{\partial h}{\partial x_3}$$



Permeability test



Constant head

$$q = -ki = -k\frac{\partial h}{\partial s} = -k\frac{\Delta h}{\Delta l}$$

$$Q = q\frac{A}{t} = -k\frac{A}{t}\frac{\Delta h}{\Delta l}$$

$$\frac{\kappa\,\gamma_w}{\mu} = |k| = \frac{Qt\Delta l}{A\Delta h} = \frac{Qt}{iA}$$

q-specific discharge [(m^3/s)/m^2]

Q-total volume(!) of water

k-coefficient of permeability

Falling head

$$Q = -k\frac{A}{t}\frac{\Delta h}{\Delta l}$$

$$Q = A_2(\Delta h_1 - \Delta h_2)$$

$$\frac{\partial Q}{\partial t} = A_2\frac{\partial h}{\partial t}$$

$$A_2\frac{\partial h}{\partial t} = -kA\frac{\Delta h}{\Delta l}$$

$$h(t) = h(0)\exp(-k\frac{At}{A_2\Delta l})$$

$$k = \frac{A_2\Delta l}{A\,t}\ln\left(\frac{h_0}{h}\right)$$

Soil type	k[m^3/s/m^2]
clay	10^{-10}-10^{-8}
silt	10^{-8}-10^{-6}
sand	10^{-6}-10^{-3}
gravel	10^{-3}-10^{-1}

Wroclaw University of Science and Technology

Groundwater flow

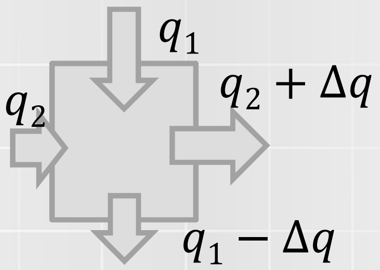
Darcy's Law

$$q_1 = -k\frac{\partial h}{\partial x_1}$$

$$q_2 = -k\frac{\partial p}{\partial x_2}$$

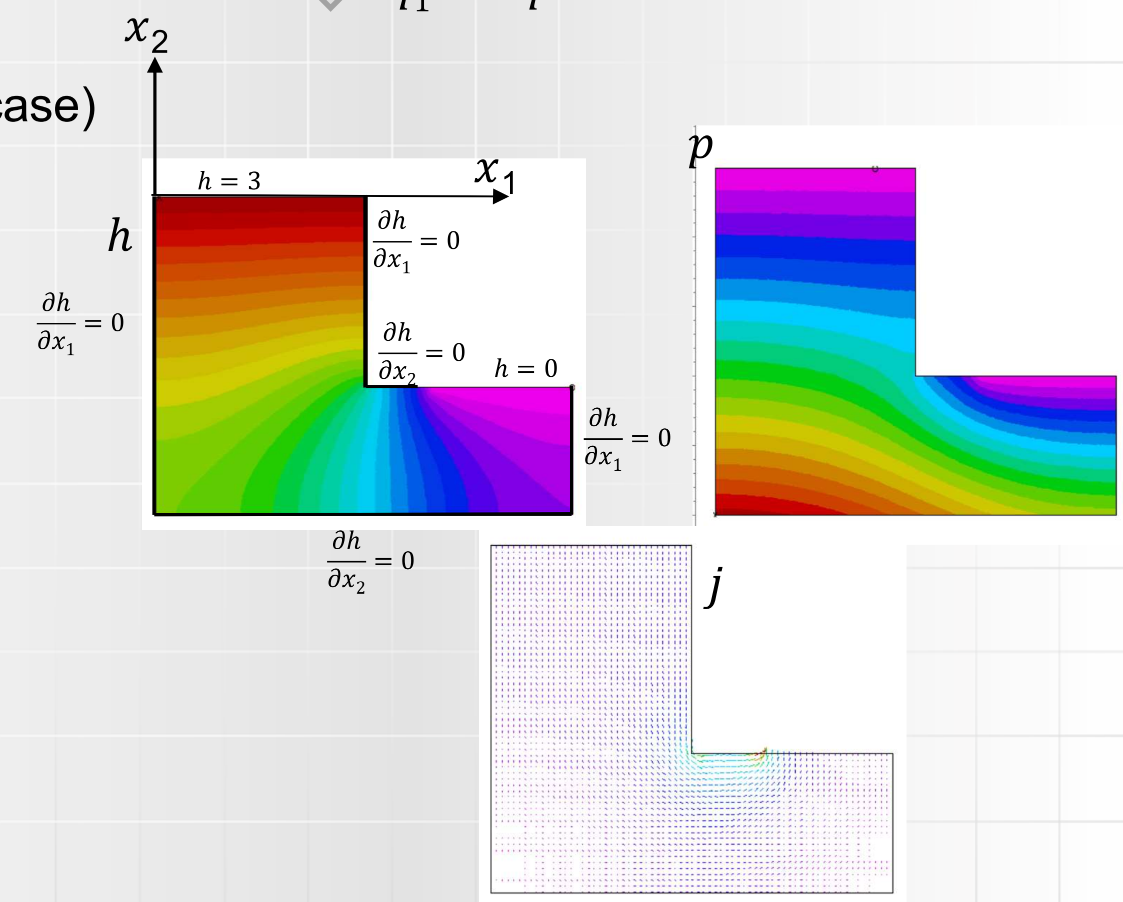
Principle of mass conservation

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0$$



Laplace equation (isotropic case)

$$-k\left(\frac{\partial^2 h}{\partial {x_1}^2} + \frac{\partial^2 h}{\partial {x_2}^2}\right) = 0$$



Upward flow

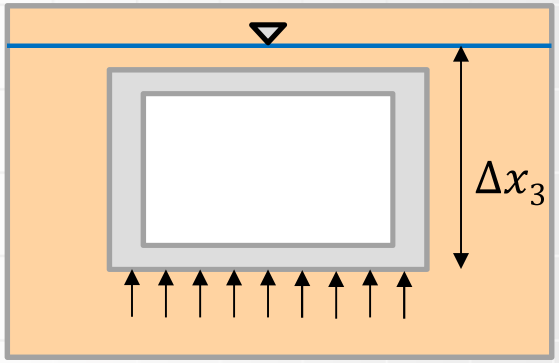
$$q = -k\frac{\partial h}{\partial s} = -k\,i \qquad j = \gamma_w \frac{\partial h}{\partial s} = \gamma_w\, i$$

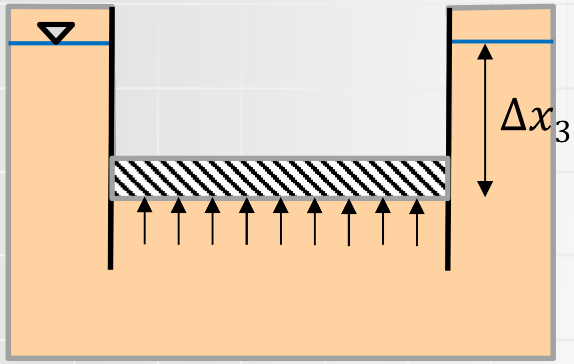
$$\sigma' = \gamma(-x_2) - \gamma_w(-x_2) - \gamma_w\, i(-x_2)$$

$$i_{cr} = \frac{\gamma - \gamma_w}{\gamma_w}$$

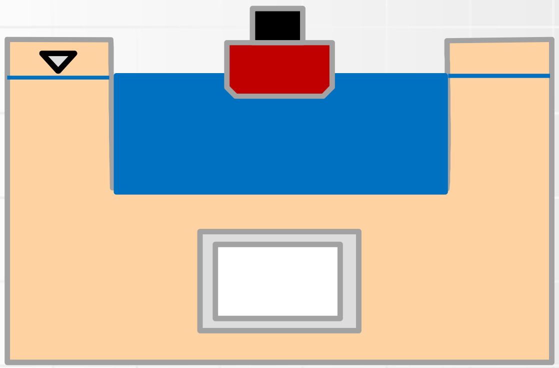
Wrocław University of Science and Technology

Flotation. Uplift.





Designing the tunel should we account for weight of the ship?



Except for the uplift are the load on slab any different?

Flow net

Darcy's Law

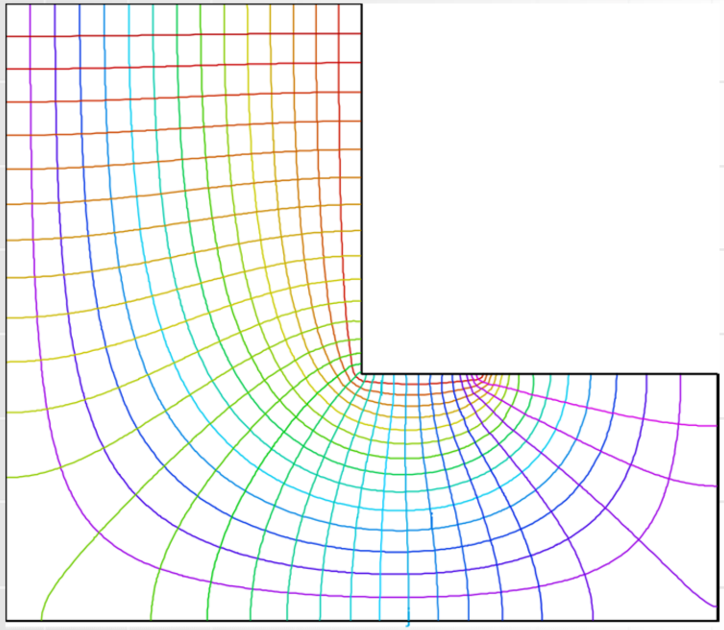
$$q_1 = -k\frac{\partial h}{\partial x_1}$$

$$q_2 = -k\frac{\partial p}{\partial x_2}$$

Principle of mass conservation

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0$$

$$-k\left(\frac{\partial^2 h}{\partial {x_1}^2} + \frac{\partial^2 h}{\partial {x_2}^2}\right) = 0$$



Potential function

$$\Phi = kh$$

$$q_1 = -\frac{\partial \Phi}{\partial x_1}$$

$$q_2 = -\frac{\partial \Phi}{\partial x_2}$$

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0$$

$$\frac{\partial^2 \Phi}{\partial {x_1}^2} + \frac{\partial^2 \Phi}{\partial x_2 2} = 0$$

Stream function

$$q_1 = -\frac{\partial \psi}{\partial x_2}$$

$$q_2 = \frac{\partial \psi}{\partial x_1}$$

$$\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = \frac{\partial^2 \psi}{\partial x_1 \partial x_2} - \frac{\partial^2 \psi}{\partial x_2 \partial x_1} \equiv 0$$

$$\frac{\partial^2 \psi}{\partial {x_1}^2} + \frac{\partial^2 \psi}{\partial x_2 2} = 0$$

Seepage force

$$j_1 = -\gamma_w \frac{\partial h}{\partial x_1} \qquad j_2 = -\gamma_w \frac{\partial h}{\partial x_2}$$

Wrocław University of Science and Technology

Richards. Van Genuchten.

Darcy's Law

$$q_1 = -k\frac{\partial h}{\partial x_1} \qquad \frac{\partial}{\partial x_1}\left(k\frac{\partial h}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k\frac{\partial h}{\partial x_2}\right) = 0$$

$$q_2 = -k\frac{\partial p}{\partial x_2} \qquad h = x_2 + \frac{p}{\gamma_w}$$

Richards equation (steady state flow)

$$\frac{\partial}{\partial x_1}\left(k(h)\frac{\partial h}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k(h)\frac{\partial h}{\partial x_2}\right) + \frac{\partial k(h)}{\partial x_2} = 0 \qquad h = \frac{p}{\gamma_w}$$

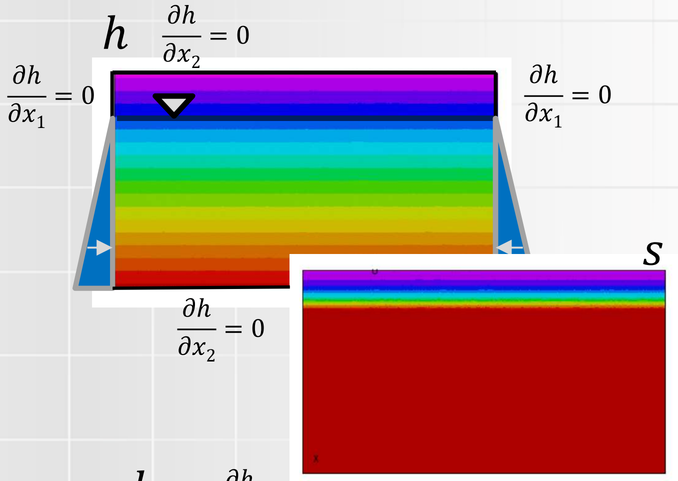
Van Genuchten model

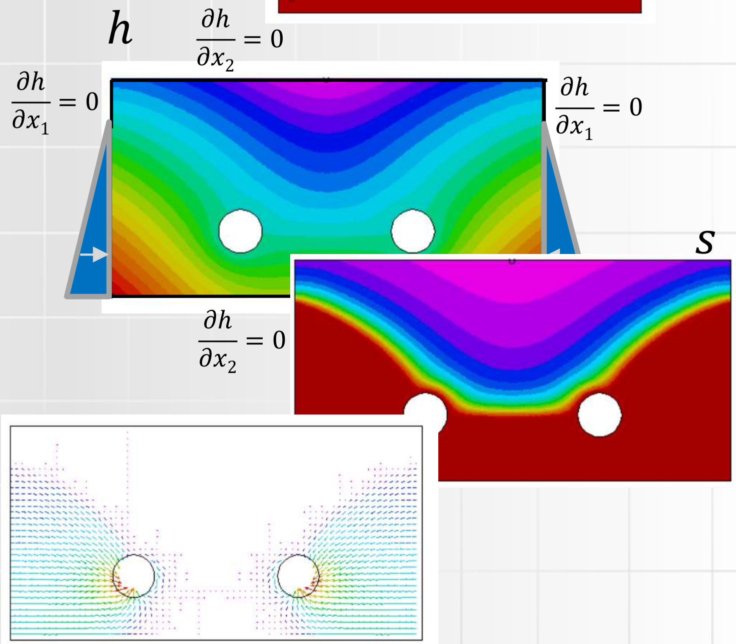
$$s(h) = H(h)\left\{s_r + \frac{s_s - s_r}{\left[1 + (\alpha|h|)^{m}\right]^{\left(1-\frac{1}{m}\right)}}\right\}$$

$$k(h) = ks\left(\frac{s(h) - s_r}{1 - s_r}\right)^3 \qquad s_r \approx 0, ss \approx 1$$

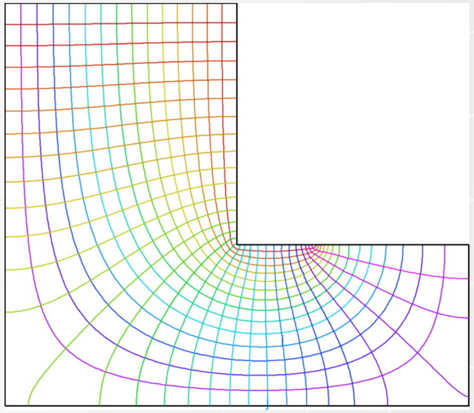
Richards equation for time dependent flow

$$n\frac{\partial s}{\partial t} + \frac{\partial}{\partial x_1}\left(k(h)\frac{\partial h}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k(h)\frac{\partial h}{\partial x_2}\right) + \frac{\partial k(h)}{\partial x_2} = 0$$





Questions



•Flow under a thin wall

•Flow under a 2-dimentional (thick) object

•Flotation of the tunnel

•Uplift

•Solving some numerical problems of steady state flow

Bibliography

Verruijt, A., & Van Baars, S. (2007). *Soil mechanics* (pp. 19-25). Delft, the Netherlands: VSSD.

Truty, A., & Podleś, K. (2009). Z_Soil. *PC Manual*.

https://en.wikipedia.org/wiki/Cauchy_stress_tensor

https://www.google.com/search?client=firefox-b&q=rotation+matrix

https://en.wikipedia.org/wiki/Richards_equation

http://www.tajnikigeotechniki.pl/